§7.2 4-manifolds with $Rm \geq 0$

- Lie algebra square

$$Rm^{\#} : \Lambda^2 M^n \longrightarrow \Lambda^2 M^n$$

$$(Rm^{\#})_{\alpha\beta} := C_{\alpha}^{\gamma\delta} C_{\beta}^{\varepsilon\zeta} Rm_{\gamma\varepsilon} Rm_{\delta\zeta}$$

then

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm^2 + Rm^{\#} \qquad (\star)$$

- 4-manifolds $\Lambda^2 = \overset{+1}{\Lambda^2_+} \oplus \Lambda^2_-$ (−1)-espace of $*$

$$Rm = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}$$

1st Bianchi id $\iff$ $tr(A) = tr(C)$

$\alpha = 1, 2, 3.$

diagonal decomposition of A, B, C

eigenvalues a_α b_α c_α

$\longrightarrow$ self adj, non-ive

A, BB^t, B^tB, C

eigenvectors φ_α^+ ψ_α^+ ψ_α^- φ_α^-

$$B\,\psi_\alpha^- = b_\alpha \psi_\alpha^+ \qquad 0 \leq b_1 \leq b_2 \leq b_3$$

(✲) ⇒ ODE of A, B, C with variable t

$$\frac{\partial}{\partial t}A = \Delta A + A^2 + 2A^{\#} + BB^t,$$
$$\frac{\partial}{\partial t}B = \Delta B + AB + BC + 2B^{\#},$$
$$\frac{\partial}{\partial t}C = \Delta C + C^2 + 2C^{\#} + B^tB,$$

⇒ ordinary differential inequality of e-values

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$$\frac{d}{dt}a_1 \geq a_1^2 + b_1^2 + 2a_2a_3,$$
$$\frac{d}{dt}a_3 \leq a_3^2 + b_3^2 + 2a_1a_2,$$
$$\frac{d}{dt}c_1 \geq c_1^2 + b_1^2 + 2c_2c_3,$$
$$\frac{d}{dt}c_3 \leq c_3^2 + b_3^2 + 2c_1c_2,$$
$$\frac{d}{dt}(b_2 + b_3) \leq a_2b_2 + a_3b_3 + b_2c_2 + b_3c_3 + 2b_1b_2 + 2b_1b_3.$$

⇒ traces satisfies

$$\frac{d}{dt}(a + c - 2b) \geq (a_1 + c_1 + 2b)(a + c - 2b)$$

The above gives pinching estimate of a_α, a, ...
which is proven via max principle

Lemma 7.18. *Let $(M^4, g(t))$ be a solution to the Ricci flow on a closed 4-manifold with positive curvature operator. There exist constants K_1, K_2, K_3, K_4, $K_5 < \infty$ and δ_1, δ_2, $\delta_3 > 0$ depending only on the initial metric $g(0)$ such that we have the following estimates for the components A, B, C of* Rm.

(1) *B pinching:*
$$(b_2 + b_3)^2 \leq K_1a_1c_1.$$
(2) *A and C pinching:*
$$a_3 \leq K_2a_1,$$
$$c_3 \leq K_2c_1.$$
(3) *B improved pinching:*
$$(b_2 + b_3)^{2+\delta_1} \leq K_3a_1c_1(a + c - 2b)^{\delta_1},$$
$$(b_2 + b_3)^{2+\delta_2} \leq K_4a_1c_1.$$
(4) *A and C improved pinching:*
$$a_3 - a_1 \leq K_5a_1^{1-\delta_3},$$
$$c_3 - c_1 \leq K_5c_1^{1-\delta_3}.$$

(3) + (4) gives control of evalues of A, B, C

$\Rightarrow$ (7.3) in Prop 7.4 (control on $|\mathring{Rm}| \leq K R^{1-\varepsilon}$)

$\Rightarrow$ Thm 7.15. ($|\mathring{Rm}| = 0$)

Thm 7.15

- (M^4, g_0)

closed, +ive sec curvature

- $\exists!$ $g(t)$ sol. of NRF $g(0) = g_0$ $\forall t \in [0, \infty)$
- as $t \to \infty$, $g(t) \xrightarrow{C^k} g_\infty$
- $M \cong S^4$ or $\underbrace{\mathbb{RP}^4}_{S^4/\mathbb{Z}_2}$

Ex. 7.19 $Rm > 0$ closed M^4

$\exists\, C < \infty$, $\delta > 0$ s.t.

$$\left| Rm - \frac{R}{24} g^2 \right| \leq C R^{2-\delta}$$

on $M^4 \times [0, T)$, $T < \infty$ singularity

$n = 4$ $2n(n-1) = 24$ given by Prop 7.4

§ 7.3 manifolds with nonnegative $Rm \geq 0$

Thm 7.15 $\longrightarrow$ Thm 7.20 $Rm \geq 0$ classification of $(\tilde{M}^4, \tilde{g}(t))$

$(\tilde{M}^4, \tilde{g}(t))$	Lie subalg of $Im(Rm) \cong \Lambda^2$
$(\mathbb{R}^4, g_{euc})$	$\{0\}$
$(S^3, h(t)) \times \mathbb{R}$	$\underline{so}(3)$
$(S^2, h_1(t)) \times (S^2, h_2(t))$	$\underline{so}(2) \times \underline{so}(2)$
$(S^2, h_1(t)) \times (\mathbb{R}^2, g_{euc})$	$\underline{so}(2)$
$\mathbb{C}P^2$	$\underline{u}(2) = \underline{so}(3) \times \underline{so}(2)$
S^4	$\underline{so}(4)$